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Positive Solutions of Three-Point Boundary Value Problems for the One-Dimensional p-Laplacian with Infinitely Many Singularities

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Abstract—We consider the singular three-point boundary value problems

$$(\phi_p(y'))' + a(t)f(y(t)) = 0, \qquad 0 < t < 1,$$
$$y'(0) = 0, \quad y(1) = \beta y(\eta),$$

where $\phi_p(s) = |s|^{p-2}s$, $p \geq 2$, $0 < \beta < 1$, $0 < \eta < 1$, $f \in C([0, +\infty), [0, +\infty))$, $a : [0,1] \to [0, +\infty)$, and has countably many singularities in $[0, 1/2)$. We show that there exist countably many positive solutions by using the fixed-point index theory. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

We are interested in the existence of positive solutions of the following singular three-point boundary value problems for one-dimensional p-Laplacian:

$$(\phi_p(y'))' + a(t)f(y(t)) = 0, \qquad 0 < t < 1, \tag{1.1}$$
$$y'(0) = 0, \quad y(1) = \beta y(\eta), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p \geq 2$, $0 < \beta < 1$, $0 < \eta < 1$, $f \in C([0, +\infty), [0, +\infty))$, $a : [0,1] \to [0, +\infty)$, and has countably many singularities in $[0, 1/2)$.

Recently, for the existence problems of positive solutions of multipoint boundary value problems for second-order ordinary differential equations, some authors have obtained the existence results. For details, see, for example, [1–5]. However, the multipoint boundary value problems treated in the above-mentioned references do not discuss problems with singularities. For the singular case of multipoint boundary value problems, with the author's knowledge, no one has studied the

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existence of positive solutions in the case. Very recently, Kaufmann and Kosmatov [6] showed that there exist countably many positive solutions for the two-point boundary value problems with infinitely many singularities of following form:

$$u''(t) + b(t)f(u(t)) = 0, \qquad 0 < t < 1, \tag{1.3}$$

$$u(0) = u(1) = 0, \tag{1.4}$$

where $b \in L^p[0,1]$ for some $p \geq 1$ and has countably many singularities in $[0, 1/2)$.

Motivated by the result of [6], we show that problems (1.1),(1.2) have infinitely many solutions if a and f satisfy some suitable conditions. The key tool in our approach is the following fixed-point index theorem [7].

THEOREM A. *(See [7].) Let E be Banach a space and $P \subset E$ be a cone in E. Let $r > 0$ define $\Omega_r = \{x \in P \mid \|x\| < r\}$. Assume that $A : \bar{\Omega} \to P$ is a completely continuous operator such that $Ax \neq x$ for $x \in \partial\Omega_r$.*

(i) *If $\|Au\| \leq \|u\|$ for $u \in \partial\Omega_r$, then*

$$i(A, \Omega_r, P) = 1.$$

(ii) *If $\|Au\| \geq \|u\|$ for $u \in \partial\Omega_r$, then*

$$i(A, \Omega_r, P) = 0.$$

From now on, in the rest of the paper, we always assume that $a(t)$ satisfies the following conditions.

(H_0) There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_{i+1} < t_i$ $(i \in N)$, $t_1 < 1/2$, $\lim_{i\to\infty} t_i = t^* \geq 0$, $\lim_{t\to t_i} a(t) = +\infty$ for all $i = 1, 2, \dots,$ and

$$0 < \int_0^1 a(s)\, ds < +\infty.$$

Moreover, $a(t)$ does not vanish identically on any subinterval of $[0,1]$.

Note, it is easy to check that Condition (H_0) implies that

$$0 < \int_0^1 \phi_q\left(\int_0^s a(s_1)\, ds_1\right) ds < +\infty,$$

where $\phi_q(s)$ is the inverse function to $\phi_p(s)$, a.e., $\phi_q(s) = |s|^{q-2}s$, $1/p + 1/q = 1$.

2. MAIN RESULT AND REMARK

Let $E = C[0,1]$, and only the supnorm is used. Denote

$$C_0^+[0,1] = \left\{y \in E : \min_{t\in[0,1]} y(t) \geq 0,\ y'(0) = 0,\ y(1) = \beta y(\eta)\right\},$$

$$P = \left\{y \in C_0^+[0,1] : y(t) \text{ is a nonnegative concave function}\right\}.$$

It is obvious that P is a cone in E.

LEMMA 2.1. *Suppose* (H_0) *holds. Then, the function*

$$A(t) = \frac{\beta}{1-\beta}\int_t^{1-t_1} \phi_q\left(\int_t^s a(s_1)\,ds_1\right)ds + \int_{t_1}^t \phi_q\left(\int_{t_1}^s a(s_1)\,ds_1\right)ds, \qquad t\in[t_1,1-t_1],$$

is continuous and positive on $[t_1, 1-t_1]$. *Furthermore,* $L(t_1) = \min_{t\in[t_1,1-t_1]} A(t) > 0$.

PROOF. At first, it is easily seen that $A(t)$ is continuous on $[t_1, 1-t_1]$. Next, let

$$A_1(t) = \frac{\beta}{1-\beta}\int_t^{1-t_1} \phi_q\left(\int_t^s a(s_1)\,ds_1\right)ds \qquad \text{and} \qquad A_2(t) = \int_{t_1}^t \phi_q\left(\int_{t_1}^s a(s_1)\,ds_1\right)ds.$$

Then, from Condition (H_0), we know that the function $A_1(t)$ is strictly monotone decreasing on $[t_1, 1-t_1]$ and $A_1(1-t_1) = 0$, the function $A_2(t)$ is strictly monotone increasing on $[t_1, 1-t_1]$ and $A_2(t_1) = 0$. So the function $A(t) = A_1(t) + A_2(t_1)$ is positive on $[t_1, 1-t_1]$, which implies $L(t_1) = \min_{t\in[t_1,1-t_1]} A(t) > 0$.

LEMMA 2.2. *Let* $y \in P$ *and* $\mu \in (0, 1/2)$. *Then,*

$$y(t) \ge \mu\|y\|, \qquad t\in[\mu, 1-\mu],$$

where $\|y\| = \sup_{t\in[0,1]} y(t)$.

PROOF. Let

$$\tau = \inf\left\{\xi\in[0,1] : \sup_{t\in[0,1]} y(t) = y(\xi)\right\}.$$

CASE (i). $\tau \in [0, \mu]$. It follows from the concavity of $y(t)$ that each point on chord between $(\tau, y(\tau))$ and $(1, y(1))$ is below the graph of $y(t)$. Thus,

$$y(t) \ge y(\tau) + \frac{y(1)-y(\tau)}{1-\tau}(t-\tau), \qquad t\in[\mu, 1-\mu].$$

Hence,

$$\begin{aligned} y(t) &\ge \min_{t\in[\mu,1-\mu]}\left[y(\tau) + \frac{y(1)-y(\tau)}{1-\tau}(t-\tau)\right] \\ &= y(\tau) + \frac{y(1)-y(\tau)}{1-\tau}(1-\mu-\tau) \\ &= \frac{1-\mu-\tau}{1-\tau}y(1) + \frac{\mu}{1-\tau}y(\tau) \ge \mu y(\tau), \end{aligned}$$

which implies that

$$y(t) \ge \mu\|y\|.$$

CASE (ii). $\tau \in [\mu, 1-\mu]$. If $t\in[\mu,\tau]$, similarly, we have

$$y(t) \ge y(\tau) + \frac{y(\tau)-y(0)}{\tau}(t-\tau), \qquad t\in[\mu,\tau].$$

Thus,

$$\begin{aligned} y(t) &\ge \min_{t\in[\mu,\tau]}\left[y(\tau) + \frac{y(\tau)-y(0)}{\tau}(t-\tau)\right] \\ &= y(\tau) + \frac{y(\tau)-y(0)}{\tau}(\mu-\tau) \\ &= \frac{\mu}{\tau}y(\tau) + \left(1-\frac{\mu}{\tau}\right)y(0) \ge \mu y(\tau). \end{aligned}$$

If $t \in [\tau, 1-\mu]$, similarly,

$$y(t) \geq y(\tau) + \frac{y(1)-y(\tau)}{1-\tau}(t-\tau), \qquad t \in [\tau, 1-\mu].$$

Thus,

$$\begin{aligned} y(t) &\geq \min_{t\in[\tau,1-\mu]} \left[y(\tau) + \frac{y(1)-y(\tau)}{1-\tau}(t-\tau) \right] \\ &= y(\tau) + \frac{y(1)-y(\tau)}{1-\tau}(1-\mu-\tau) \\ &= \frac{\mu}{1-\tau} y(\tau) + \frac{1-\mu-\tau}{1-\tau} y(1) \geq \mu y(\tau). \end{aligned}$$

Therefore, we obtain

$$y(t) \geq \mu\|y\|, \qquad t \in [\mu, 1-\mu].$$

CASE (iii). $\tau \in [1-\mu, 1]$. Similarly, we have

$$y(t) \geq y(\tau) + \frac{y(\tau)-y(0)}{\tau}(t-\tau), \qquad t \in [\mu, 1-\mu].$$

Thus,

$$\begin{aligned} y(t) &\geq \min_{t\in[\mu,1-\mu]} \left[y(\tau) + \frac{y(\tau)-y(0)}{\tau}(t-\tau) \right] \\ &= y(\tau) + \frac{y(\tau)-y(0)}{\tau}(\mu-\tau) \\ &= \frac{\mu}{\tau} y(\tau) + (1-\frac{\mu}{\tau}) y(0) \geq \mu y(\tau), \end{aligned}$$

which yields

$$y(t) \geq \mu\|y\|, \qquad t \in [\mu, 1-\mu].$$

This completes the proof.

Now we define an operator $T : P \to P$ by

$$(Ty)(t) = \frac{\beta}{1-\beta}\int_\eta^1 \phi_q\left(\int_0^s a(s_1)f(x(s_1))\,ds_1\right)ds + \int_t^1 \phi_q\left(\int_0^s a(s_1)f(x(s_1))\,ds_1\right)ds. \quad (2.1)$$

Then, it is easy to see that $(Ty)(t) \geq 0$ $(0 \leq t \leq 1)$, $(Ty)'(0) = 0$, $(Ty)(1) = \beta(Ty)(\eta)$, and $(Ty)'(t) = -\phi_q(\int_0^t a(s)f(x(s))\,ds) \leq 0$, $(\phi_p(Ty)'(t))' = -a(t)f(y(t)) \leq 0$. This shows that $T(P) \subset P$.

LEMMA 2.3. *Let* (H_0) *hold, then $T : P \to P$ is completely continuous.*

PROOF. Let D be a bounded subset of P and $M > 0$ is the constant such that $\|y\| \leq M$ for $y \in D$. Then, we

$$\|Ty\| \leq \left[\frac{1}{1-\beta}\int_0^1 \phi_q\left(\int_0^s a(s_1)\,ds_1\right)ds\right]\phi_q(\sup\{f(y) : y \in D\}),$$

which implies the boundedness of $T(D)$.

Now, for any $y \in D$, $0 \leq t_1 < t_2 \leq 1$, we have

$$\begin{aligned} |(Ty)(t_1) - (Ty)(t_2)| &= \left|\int_{t_1}^{t_2} \phi_q\left(\int_0^s a(s_1)f(x(s_1))\,ds_1\right)ds\right| \\ &\leq \phi_q\left(\int_0^1 a(s)\,ds\right)\phi_q(\sup\{f(y) : y \in D\})|t_1 - t_2|. \end{aligned}$$

Therefore, $T(D)$ is equicontinuous.

Finally, in view if the continuity of f and the Lebesgue's dominated convergence theorem, it is easy to see that T is continuous on D. Thus, the Arzela-Ascoli theorem implies that $T : P \to P$ is completely continuous.

THEOREM 2.1. *Assume that Condition* (H_0) *holds. Let* $\{\mu_k\}_{k=1}^{\infty}$ *such that* $\mu_k \in (t_{k+1}, t_k)$ $(k = 1, 2, \dots)$. *Let* $\{R_k\}_{k=1}^{\infty}$ *and* $\{r_k\}_{k=1}^{\infty}$ *such that*

$$R_{k+1} < \mu_k r_k < r_k < \Lambda_1 r_k < R_k, \qquad k = 1, 2, \dots,$$

where $\Lambda_1 \in (1/L(t_1), +\infty)$.

Furthermore, for each natural number k, *assume that* f *satisfies*

(H_1) $f(u) \geq (\Lambda_1 r_k)^{p-1}$ *for all* $u \in [\mu_k r_k, r_k]$;

(H_2) $f(u) \leq (\Lambda_2 R_k)^{p-1}$ *for all* $u \in [0, R_k]$, *where* $0 < \Lambda_2 < (((1-\beta\eta)/(1-\beta))\phi_q(\int_0^1 a(s)\,ds))^{-1}$.

Then, the problems (1.1),(1.2) have infinitely many positive solutions $\{u_i\}_{i=1}^{\infty}$ *such that*

$$r_i \leq \|u_i\| \leq R_i, \qquad \text{for each } i = 1, 2, \dots.$$

PROOF. Since $t^* < t_{k+1} < \mu_k < t_k < 1/2$ for all $k \in N$. Then, for each $k \in N$ and $y \in P$, from Lemma 2.2, we have

$$y(t) \geq \mu_k \|y\|, \qquad t \in [\mu_k, 1 - \mu_k]. \tag{2.2}$$

Consider the sequence $\{\Omega_k^1\}_{k=1}^{\infty}$ and $\{\Omega_k^2\}_{k=1}^{\infty}$ of open subsets of $C_0^+[0,1]$ defined by

$$\Omega_k^1 = \{y \in P : \|y\| < r_k\},$$
$$\Omega_k^2 = \{y \in P : \|y\| < R_k\}.$$

Now, fix k and let $y \in \partial\Omega_k^1$. From (2.2), we have $r_k = \|y\| \geq y(s) \geq \mu_k\|y\| = \mu_k r_k$, for $s \in [\mu_k, 1-\mu_k]$. In the following, we consider three cases (note that $[t_1, 1-t_1] \subset [\mu_k, 1-\mu_k]$).

CASE 1. If $\eta \in [t_1, 1-t_1]$. In this case, from (2.1), Condition (H_1), and Lemma 2.1, we get

$$\begin{aligned}
\|Ty\| &= (Ty)(0) \\
&= \frac{\beta}{1-\beta}\int_\eta^1 \phi_q\left(\int_0^s a(s_1) f(y(s_1))\,ds_1\right) ds + \int_0^1 \phi_q\left(\int_0^s a(s_1) f(y(s_1))\,ds_1\right) ds \\
&\geq \frac{\beta}{1-\beta}\int_\eta^{1-t_1} \phi_q\left(\int_\eta^s a(s_1) f(y(s_1))\,ds_1\right) ds + \int_{t_1}^\eta \phi_q\left(\int_{t_1}^s a(s_1) f(y(s_1))\,ds_1\right) ds \\
&\geq \Lambda_1 r_k\left[\frac{\beta}{1-\beta}\int_\eta^{1-t_1} \phi_q\left(\int_\eta^s a(s_1)\,ds_1\right) ds + \int_{t_1}^\eta \phi_q\left(\int_{t_1}^s a(s_1)\,ds_1\right) ds\right] \\
&\geq \Lambda_1 r_k A(\eta) \\
&\geq \Lambda_1 r_k L(t_1) > r_k = \|y\|.
\end{aligned}$$

CASE 2. If $\eta \in (0, t_1)$, from (2.1), Condition (H_1), and Lemma 2.1, we get

$$\begin{aligned}
\|Ty\| &= (Ty)(0) \\
&= \frac{\beta}{1-\beta}\int_\eta^1 \phi_q\left(\int_0^s a(s_1) f(y(s_1))\,ds_1\right) ds + \int_0^1 \phi_q\left(\int_0^s a(s_1) f(y(s_1))\,ds_1\right) ds \\
&\geq \frac{\beta}{1-\beta}\int_{t_1}^{1-t_1} \phi_q\left(\int_{t_1}^s a(s_1) f(y(s_1))\,ds_1\right) ds \\
&\geq \Lambda_1 r_k A(t_1) \\
&\geq \Lambda_1 r_k L(t_1) > r_k = \|y\|.
\end{aligned}$$

CASE 3. If $\eta \in (1-t_1, 1)$, from (2.1), Condition (H_1), and Lemma 2.1, we get

$$\begin{aligned}
\|Ty\| &= (Ty)(0) \\
&= \frac{\beta}{1-\beta}\int_\eta^1 \phi_q\left(\int_0^s a(s_1) f(y(s_1))\,ds_1\right) ds + \int_0^1 \phi_q\left(\int_0^s a(s_1) f(y(s_1))\,ds_1\right) ds \\
&\geq \int_{t_1}^{1-t_1} \phi_q\left(\int_{t_1}^s a(s_1) f(y(s_1))\,ds_1\right) ds \\
&\geq \Lambda_1 r_k A(1-t_1) \\
&\geq \Lambda_1 r_k L(t_1) > r_k = \|y\|.
\end{aligned}$$

Thus, in all cases, an application of Theorem A shows that

$$i\left(T, \Omega_k^1, P_k\right) = 0. \tag{2.3}$$

Next, let $y \in \partial\Omega_k^2$. Then, from (2.1), Condition (H_2), we have

$$\begin{aligned}
\|Ty\| &= (Ty)(0) \\
&= \frac{\beta}{1-\beta}\int_\eta^1 \phi_q\left(\int_0^s a(s_1)f(y(s_1))\,ds_1\right) ds + \int_0^1 \phi_q\left(\int_0^s a(s_1)f(y(s_1))\,ds_1\right) ds \\
&\le \Lambda_2 R_k\left[\frac{\beta}{1-\beta}\int_\eta^1 \phi_q\left(\int_0^s a(s_1)\,ds_1\right) ds + \int_0^1 \phi_q\left(\int_0^s a(s_1)\,ds_1\right) ds\right] \\
&\le \Lambda_2 R_k \cdot \frac{1-\beta\eta}{1-\beta}\phi_q\left(\int_0^1 a(s)\,ds\right) \\
&< R_k = \|y\|.
\end{aligned}$$

Thus, Theorem A implies

$$i\left(T, \Omega_K^2, P_k\right) = 1. \tag{2.4}$$

Hence, since $r_k < R_k$ for $k \in N$, (2.3) and (2.4), it follows from additivity of the fixed-point index that

$$i\left(T, \Omega_k^2 \setminus \bar{\Omega}_k^1, P\right) = 1, \qquad \text{for } k \in N.$$

Thus, for each $k \in N$, T has a fixed point in $\Omega_k^2 \setminus \bar{\Omega}_k^1$ such that $r_k \le \|y\| \le R_k$. Since $k \in N$ was arbitrary, the proof is complete.

REMARK. There exist many functions $a(t)$ that satisfy Condition (H_0). For example, let $\Delta = \sqrt{2}(\pi^2/3 - 9/4)$, and define

$$t_0 = \frac{5}{16}, \quad t_n = t_0 - \sum_{i=0}^{n-1}\frac{1}{(i+2)^4}, \qquad n = 1, 2, \dots.$$

We consider the function $a(t) : [0,1] \to (0, +\infty)$ given by

$$a(t) = \sum_{n=1}^{\infty} a_n(t), \qquad t \in [0,1],$$

where

$$a_n(t) = \begin{cases} \dfrac{2}{n(n+1)(t_{n+1}+t_n)}, & 0 \le t < \dfrac{t_{n+1}+t_n}{2}, \\[2mm] \dfrac{1}{\Delta(t_n - t)^{1/2}}, & \dfrac{t_{n+1}+t_n}{2} \le t < t_n, \\[2mm] \dfrac{1}{\Delta(t-t_n)^{1/2}}, & t_n < t \le \dfrac{t_n + t_{n-1}}{2}, \\[2mm] \dfrac{2}{n(n+1)(2-t_n-t_{n-1})}, & \dfrac{t_n+t_{n-1}}{2} < t \le 1. \end{cases}$$

At first, it is easily seen that $t_1 = 1/4 < 1/2$, $t_n - t_{n+1} = 1/((n+2)^4)$, $n = 1, 2, \dots$, and (note that $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$)

$$t^* = \lim_{n\to\infty} t_n = \frac{5}{16} - \sum_{i=0}^{\infty}\frac{1}{(i+2)^4} = \frac{5}{16} - \left(\frac{\pi^4}{90} - 1\right) = \frac{21}{16} - \frac{\pi^4}{90} > \frac{1}{5}.$$

Next, since $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_0^1 a_n(t)\,dt &= \sum_{n=1}^{\infty} \frac{2}{n(n+1)} + \frac{1}{\Delta} \sum_{n=1}^{\infty} \left[\int_{(t_{n+1}+t_n)/2}^{t_n} \frac{1}{(t_n - t)^{1/2}}\,dt \right. \\
&\quad \left. + \int_{t_n}^{(t_n+t_{n-1})/2} \frac{1}{(t-t_n)^{1/2}}\,dt \right] \\
&= 2 + \frac{\sqrt{2}}{\Delta} \sum_{n=1}^{\infty} \left[(t_n - t_{n+1})^{1/2} + (t_{n-1} - t_n)^{1/2} \right] \\
&= 2 + \frac{\sqrt{2}}{\Delta} \sum_{n=1}^{\infty} \left[\frac{1}{(n+2)^2} + \frac{1}{(n+1)^2} \right] \\
&= 2 + \frac{\sqrt{2}}{\Delta} \left[\left(\frac{\pi^2}{6} - \frac{5}{4} \right) + \left(\frac{\pi^2}{6} - 1 \right) \right] \\
&= 2 + \frac{\sqrt{2}}{\Delta} \left[\frac{\pi^2}{3} - \frac{9}{4} \right] \\
&= 3.
\end{aligned}$$

Hence,

$$\int_0^1 a(t)\,dt = \int_0^1 \sum_{n=1}^{\infty} a_n(t)\,dt = \sum_{n=1}^{\infty} \int_0^1 a_n(t)\,dt < \infty,$$

which implies that Condition (H_0) holds.

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